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1979 J. Phys. A: Math. Gen. 12 743

(http://iopscience.iop.org/0305-4470/12/5/020)

J. Phys. A: Math. Gen., Vol. 12, No. 5, 1979. Printed in Great Britain

COMMENT

The critical isotherm of the four-dimensional Ising model

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Received 20 December 1978

Abstract. The critical isotherm of the Ising model for the four-dimensional simple hypercubic lattice is studied using a high-field expansion through μ^{15}. The series for the magnetisation is analysed for singularities of the asymptotic form $E(1-\mu)^{1/3}|\ln(1-\mu)|^p$, predicted by renormalisation group theory. Good convergence is obtained for values of p around $\frac{1}{3}$ (the renormalisation group prediction) and we estimate $p = 0{\cdot}30 \pm 0{\cdot}05$. Assuming $p = \frac{1}{3}$, the critical amplitude E is estimated to be $0{\cdot}896 \pm 0{\cdot}02$.

The critical isotherm of the spin-$\frac{1}{2}$ Ising model (with nearest-neighbour isotropic interactions) has been extensively studied using exact series expansions (Gaunt *et al* 1964, Gaunt 1967, Gaunt and Sykes 1972) for all the standard lattices of dimensionality $d = 2$ and 3. The magnetisation M has the expansion

$$M(u, \mu) = 1 - 2\sum_{s=1}^{\infty} sL_s(u)\mu^s \tag{1}$$

where the temperature and magnetic field variables are defined, using the conventional notation, by

$$u = \exp(-4J/kT), \qquad \mu = \exp(-2mH/kT). \tag{2}$$

Evaluating the high-field polynomials $L_s(u)$ at the critical temperature $u = u_c = \exp(-4J/kT_c)$ yields an expansion for $M(u_c, \mu) \equiv M(\mu)$ in powers of μ. In the most recent study, Gaunt and Sykes (1972) improved upon earlier work by using longer series (up to μ^{21} for $d = 2$ and μ^{17} for $d = 3$), and, for the three-dimensional lattices, more reliable estimates of the critical temperatures. The latter were obtained by analysing high-temperature series for the zero-field susceptibility χ_0 in powers of $v = \tanh(J/kT)$. Assuming a singularity of the asymptotic form

$$M \sim E(1-\mu)^{1/\delta} \qquad (T = T_c, H \to 0+) \tag{3}$$

they estimated the critical exponent δ to be

$$\begin{aligned} \delta &= 15{\cdot}00 \pm 0{\cdot}08 \qquad (d = 2) \\ \delta &= \ \ 5{\cdot}00 \pm 0{\cdot}05 \qquad (d = 3). \end{aligned} \tag{4}$$

The central two-dimensional value has subsequently been proved to be exact by Abraham (1977). In three dimensions, renormalisation group (RG) theory predicts the much smaller value $\delta = 4{\cdot}82 \pm 0{\cdot}06$ (Baker *et al* 1978). The discrepancies between series and RG estimates of critical exponents for the $d = 3$ Ising model have been the

0305-4470/79/050743+04\$01.00

subject of much recent work (Gaunt and Sykes 1979, Bessis *et al* 1979, Nickel and Sharpe 1979).

In this work we investigate the critical isotherm for the $d=4$ Ising model. RG calculations (Brézin *et al* 1976, Larkin and Khmel'nitskii 1969) show that at T_c the critical equation of state reduces to

$$h \sim DM^3/|\ln M| \qquad (h=2mH/kT_c) \tag{5}$$

which, on inversion, yields to leading order

$$M \sim E(1-\mu)^{1/3}|\ln(1-\mu)|^{1/3} \tag{6}$$

where $D=\frac{1}{3}E^{-3}$. It appears that the dominant asymptotic behaviour should be given by (3) with $\delta=3$, but modified by a confluent logarithmic correction term of the kind characteristic of the critical dimensionality.

To study this problem we have followed the procedure already outlined for $d=2$ and 3. The high-field polynomials for the simple hypercubic (HSC) lattice have been given by Sykes (1979) through $L_{15}(u)$, and an accurate estimate of the critical temperature, namely

$$1/v_c = 6\cdot7315 \pm 0\cdot0015, \tag{7}$$

has been obtained by Gaunt *et al* (1979) from an analysis of the high-temperature expansion of $\chi_0(v)$ through v^{17}. In this way we have derived an expansion for $M(\mu)$ through order μ^{15}.

Padé approximants to $(d/d\mu)\ln M(\mu)$ show, in addition to a dominant singularity close to $\mu=1$, a pole–zero sequence along the real μ axis for $\mu>1$. This is consistent with a singularity structure more complicated than (3), possibly of the form (6). The $D\log$ Padé approximants also exhibit a pole on the real negative μ axis at $\mu=-\mu_s \simeq -2\cdot7$. This singularity which is also present for loose-packed lattices in two and three dimensions (Gaunt and Sykes 1972) is so far outside the circle of convergence $|\mu|=1$ that it does not cause too much interference.

We now analyse the series $M(\mu)$ for a singularity of the form

$$M \sim E(1-\mu)^{1/3}|\ln(1-\mu)|^{p} \tag{8}$$

using a method of analysis similar to that developed by Guttmann (1978). We eliminate essentially all interference from the singularity at $\mu=-\mu_s$ by transforming to a new variable x defined by

$$x=(\mu_s+1)\mu/(\mu_s+\mu) \qquad (\mu_s=2\cdot7). \tag{9}$$

This has the effect of mapping the singularity at $-\mu_s$ to infinity while leaving the physical singularity at $\mu=1$ unchanged. Writing the transformed series as

$$M_T(x)=\sum_{n\geq0} a_n x^n \tag{10}$$

we form the ratios $r_n=a_n/a_{n-1}$. Defining the function $f(x)$ by

$$f(x)=x^{-p^*}(1-x)^{1/3}|\ln(1-x)|^{p^*}=\sum_{n\geq0} b_n x^n \tag{11}$$

we calculate the ratios $r_n^*=b_n/b_{n-1}$ and compare the behaviour of the r_n^* for a range of values of p^* with the behaviour of the r_n. As $n\to\infty$, the sequence $R_n=r_n/r_n^*$ and the exponent estimates $n(R_n-1)$ should approach one and zero respectively, with zero slope when $p^* = p$. The expected behaviour is obtained for values of p^* in the vicinity of 0·30; a selection of our results is shown in table 1. We make the estimate

$$p = 0{\cdot}30 \pm 0{\cdot}05 \tag{12}$$

which is consistent with $p = \frac{1}{3}$, the RG prediction. The uncertainties in $1/v_c$ quoted in (7) leave (12) unchanged.

Table 1. Analysis of the transformed critical isotherm series for the $d = 4$ Ising model.

p^*	n	R_n	Linear extrapolants	Quadratic extrapolants	Exponent $n(R_n - 1)$	Linear extrapolants
0·25	8	1·003 33	0·992 33	1·015 83	0·0266	−0·0270
	9	1·002 50	0·995 91	1·008 43	0·0225	−0·0102
	10	1·002 02	0·997 63	1·004 50	0·0202	−0·0012
	11	1·001 70	0·998 52	1·002 56	0·0187	0·0039
	12	1·001 48	0·999 04	1·001 62	0·0177	0·0072
	13	1·001 31	0·999 36	1·001 12	0·0171	0·0094
	14	1.001 19	0·999 56	1·000 80	0·0167	0·0110
	15	1·001 09	0·999 70	1·000 56	0·0164	0·0121
0·30	8	0·998 68	0·995 61	1·015 47	−0·0105	−0·0413
	9	0·998 65	0·998 43	1·008 30	−0·0121	−0·0246
	10	0·998 75	0·999 64	1·004 48	−0·0125	−0·0157
	11	0·998 88	1·000 18	1·002 59	−0·0123	−0·0105
	12	0·999 01	1·000 43	1·001 67	−0·0119	−0·0072
	13	0·999 13	1·000 54	1·001 18	−0·0113	−0·0048
	14	0·999 23	1·000 59	1·000 86	−0·0107	−0·0031
	15	0·999 32	1·000 59	1·000 63	−0·0101	−0·0019
0·35	8	0·993 66	0·999 22	1·015 12	−0·0507	−0·0561
	9	0·994 50	1·001 22	1·008 19	−0·0495	−0·0397
	10	0·995 24	1·001 87	1·004 47	−0·0476	−0·0308
	11	0·995 85	1·002 00	1·002 62	−0·0456	−0·0256
	12	0·996 36	1·001 96	1·001 73	−0·0436	−0·0221
	13	0·996 79	1·001 85	1·001 24	−0·0418	−0·0196
	14	0·997 14	1·001 72	1·000 93	−0·0401	−0·0177
	15	0·997 43	1·001 58	1·000 70	−0·0385	−0·0163

To calculate the amplitude E we divide the expansion (10) of $M_T(x)$ by the expansion of $f(x)$ with $p^* = \frac{1}{3}$, form Padé approximants to the resulting series, and evaluate them at $x = 1$. This gives an estimate of $(\mu_s + 1)^{1/3}\mu_s^{-1/3}E$ from which we obtain

$$E = 0{\cdot}896 \pm 0{\cdot}02 \mp 1{\cdot}6\Delta p \mp 1{\cdot}3\Delta(v_c^{-1}). \tag{13}$$

Δz represents the change in z, and hence the last two terms in (13) measure the sensitivity of E to small changes in p and v_c^{-1}. Converting (13) to the amplitude D defined by (5) we find

$$D_{\text{HSC}} = 0{\cdot}463 \pm 0{\cdot}03. \tag{14}$$

In two and three dimensions, D is defined by inverting (3) to give

$$h \sim DM^{\delta} \qquad (D = E^{-\delta}) \tag{15}$$

where (Gaunt and Sykes 1972)

$$D_{SC} = 0{\cdot}535 \pm 0{\cdot}015, \qquad D_{SQ} = 0{\cdot}847 \pm 0{\cdot}004 \tag{16}$$

for the simple cubic and simple quadratic lattices repectively. Hence, for the simple hypercubic lattices of dimensionality $d = 2$, 3 and 4, the amplitude D appears to decrease monotonically with increasing d. It is interesting to speculate whether for large enough d this trend will be reversed and the amplitude approach the mean-field value $D_{MF} = \frac{2}{3}$.

We conclude that near the critical point the behaviour of the magnetisation as a function of the magnetic field along the critical isotherm for the $d = 4$ simple hypercubic lattice is consistent with the asymptotic form predicted by RG theory. Thus, assuming $\delta = 3$, we estimate the exponent of the logarithmic correction term to be $0{\cdot}30 \pm 0{\cdot}05$, which agrees with the predicted value of $\frac{1}{3}$ to within the uncertainties of extrapolation.

This work has been supported (in part) through a Research Grant from the Science Research Council.

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